A dynamic nonlinear model for saturation in industrial growth

Arnab K. Ray*
Homi Bhabha Centre for Science Education, TIFR, V. N. Purav Marg, Mankhurd, Mumbai 400088, India

A general nonlinear logistic equation has been proposed to model long-time saturation in industrial growth. An integral solution of this equation has been derived for any arbitrary degree of nonlinearity. A time scale for the onset of nonlinear saturation in industrial growth can be estimated from an equipartition condition between nonlinearity and purely exponential growth. Precise predictions can be made about the limiting values of the annual revenue and the human resource content that an industrial organisation may attain. These variables have also been modelled to set up an autonomous first-order dynamical system, whose equilibrium condition forms a stable node (an attractor state) in a related phase portrait. The theoretical model has received close support from all relevant data pertaining to the well-known global company, *IBM*.

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In view of the current economic recession that is prevailing globally, it has become imperative to furnish mathematical models of greater degrees of quantitative accuracy to understand stagnation in economic growth, be it of states or of industrial organisations. When it comes to making forecasts about how the future might unfold for a state economy or an organisation, one realises the acute necessity for proper deterministic models. A recognition is gradually gaining currency that economic models should have robust predictive power, and this power should issue from the compatibility of the pertinent mathematical models with empirical data [1].

Addressing this requirement is the principal objective of this work, and this has been done by making a case study on industrial growth. Many aspects of industrial growth lend themselves to well-formulated mathematical analyses. The health of a company is to be judged from the revenue that it is capable of generating, as well as the extent of human resource that it is capable of employing in achieving its objectives. One could make quantitative measures of all these variables, and this makes it relatively easy to have a clear understanding of industrial growth pattern, as well as to posit a mathematical model for it. The approach to these issues here is predominantly based on the use of standard mathematical tools of nonlinear dynamics and dynamical systems [2, 3, 4]. There is a general appreciation that even when an industrial organisation displays noticeable (very commonly exponential) growth in the early stages, there is a saturation of this growth towards a terminal end after the elapse of a certain scale of time [5]. As the system size (reflective of the scale of operations) begins to grow through the passage of time, a self-regulatory mechanism takes effective control over growth and gradually drives the system towards a saturated terminal state. And so, on very large scales of industrial operations, a clear understanding could be derived about the constrained feature of the space within which an organisation functions.

The theory developed here has been subjected to empirical test with the help of data collected from an industrial organisation that is global in character, i.e. its presence is to be seen and felt all over the world. This choice is dictated by the requirement that one would like to understand the global growth behaviour of a company, whose operating space is by definition on the largest available scale, and, therefore, the overall pattern of its growth would be free of local inhomogeneities. To this end it has been worthwhile to study the revenue generating capacity and the growth of the human resouce strength of the multi-national company, *IBM*. This organisation has been in existence in its presently known form for nearly a century. Besides this, it has spread all over the globe. So on both of these counts, a company like *IBM* is perfectly suited for the present study. Data about its annual revenue generation, the net annual earnings and the cumulative human resource strength, dating from the year 1914, have been published on the company website[1] itself . It has been satisfying to note that, analysed according to the stated objectives and specifications of this work, the *IBM* data actually give a striking match with the mathematical models forwarded here. Both the capacity for revenue generation and the human resource content of *IBM*, over a period of more than ninety years of the existence of the company (this long period is actually quite expedient for this study, since it is concerned with the growth of an industrial organisation from its inception to its terminal stage), show an initial phase of exponential growth, to be followed later by saturated growth towards a terminal state.

A growth trend of this type can be described satisfactorily by a logistic differential equation, usually of second-degree nonlinearity, as it is done to study the growth of a population [2, 3]. Regarding the study of growth from industrial data, a preceding work [6] has pedagogically underlined the relevance of various model differential equations of increasing complexity. Along these lines, a generalisation of the logistic prescription, to any arbitrary degree of nonlinearity, is being posited here, to follow industrial growth through time, t. Such a generalised logistic equation will read as

$$\dot{\phi}(t) = \lambda\phi\left(1 - \eta\phi^{\alpha}\right) , \qquad (1)$$

where ϕ can be any relevant variable to guage the health of a

[1] `http://www-03.ibm.com/ibm/history`

firm (with the "dot" on ϕ being its simple time derivative), like its annual revenue (or cumulative revenue growth) and human resource strength. The parameters α and η are, respectively, the nonlinear saturation exponent, and the "tuning" parameter for nonlinearity. Both influence the saturation behaviour of firm growth. In the context of the growth of an industrial organisation, one can identify various factors that may contribute collectively to this saturation in growth. These factors can be both economic and non-economic in nature. Some may operate internally, while others can make their impact externally. A primary factor, regarding this study at least, is the space within which an organisation can be allowed to grow. If this space is constrained to be of a finite size (as, in a practical sense, it has to be), then, of course, terminal behaviour becomes a distinct possibility. Even as it continues to grow, an organisation will gradually have to contend with the boundaries of the space within which it has to operate. This brings growth to a slow halt. Indeed, saturation in growth due to finite-size effects is understood well by now in other situations of economic interest where physical models can be applied [7]. The adverse conditions against growth can be further aggravated by the presence of rival organisations competing for the same space. The last factor can become particularly crucial when a miscalculation is made in assessing future directions of growth vis-a-vis those of rival organisations — both the existing ones and the ones that might emerge in the future.

Integration of Eq. (1), which is a nonlinear differential equation, yields the general integral solution (for $\alpha \neq 0$),

$$\phi(t) = \left[\eta + c^{-\alpha} \exp\left(-\alpha\lambda t\right)\right]^{-1/\alpha} , \tag{2}$$

in which c is an integration constant. The fit of the foregoing integral solution with the data has been shown in Fig. 1, which gives a $\log$-$\log$ plot of the annual revenue, $R_{\rm a}$, that *IBM* has generated over time, t. Here the annual revenue has been measured in millions of dollars, and time has been scaled in years. The data and the theoretical model given by Eq. (2) agree well with each other, especially on mature time scales, where nonlinear saturation is most conspicuous. Similar features are evident in Fig. 2, which plots the *cumulative* growth of the *IBM* revenue, $R_{\rm c}$ (in millions of dollars), with each successive year, plotted as t. The prescription of the cumulative growth needs to be stressed upon here. For any evolving system, its rate of growth is almost always a direct mathematical function of its current state (which has been built up in a cumulative fashion). Very frequently this kind of dependence leads to an exponential growth pattern, something that is quite relevant here, especially as regards the early growth of *IBM*. Besides, the distribution of the cumulative revenue is much more free of fluctuations than the distribution of the annual revenue, and as such the former is better suited for modelling.

While the exponential feature may be appropriate for modelling the early stages of growth, the later stages shift into a saturation mode. A limiting value in relation to this saturated state could be found by setting $\dot{\phi} = 0$ in Eq. (1), and this will

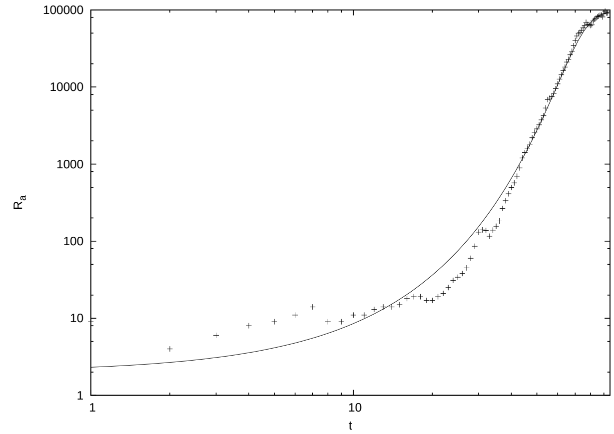

FIG. 1: The continuous curve gives the model fit for the annual revenue, $R_{\rm a}$, generated by *IBM*. The fit by the theoretical model agrees well on nonlinear time scales, for $\alpha = 1$, $\lambda = 0.145$ and $\eta = 10^{-5}$. Here $R_{\rm a}$ has been scaled in millions of dollars, while t has been scaled in years.

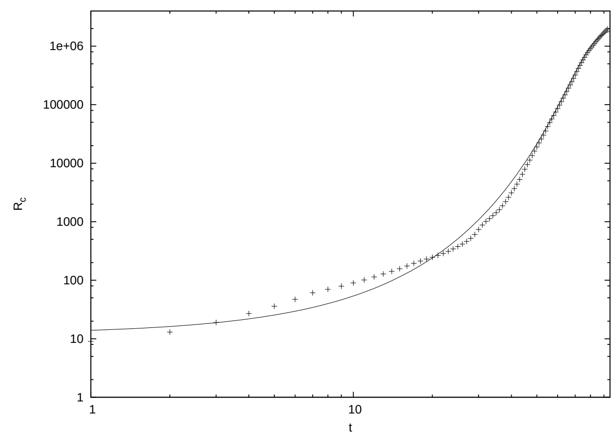

FIG. 2: The *cumulative* growth of the annual revenue, $R_{\rm c}$, generated by *IBM* is fitted by the theoretical model (given as the continuous curve). The fit is very close once again on nonlinear scales, for $\alpha = 1$, $\lambda = 0.15$ and $\eta = 5 \times 10^{-7}$. As before, $R_{\rm c}$ has been measured in millions of dollars, while t has been scaled in years. This distribution is evidently more free of fluctuations than the previous one.

lead to

$$\phi_{\rm sat} = \eta^{-1/\alpha} . \tag{3}$$

Making use of the values of α and η, which have fitted the saturation properties of the plot in Fig. 1, a prediction can be made that the maximum possible *annual* revenue (*not* the cumulative revenue plotted in Fig. 2) that *IBM* can generate will be about 100 billion dollars.

Another point of great interest is that the growth data have been fitted very well by the simplest possible case of nonlinearity, given by $\alpha = 1$. This will immediately place the present mathematical problem in the same class of the logistic

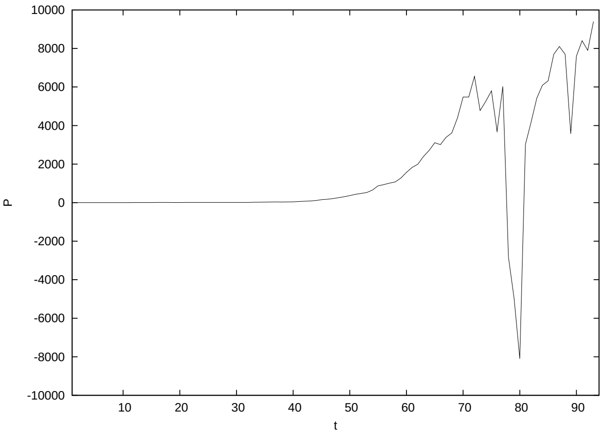

FIG. 3: The net annual earnings made by *IBM* has shown steady growth, except for the early years of the 1990s decade, which was about 80 years of the existence of the company. Around this time the company suffered major losses in its net earnings, and this time scale corresponds very closely to the time scale for the onset of nonlinear saturation in growth indicated by Eq. (4).

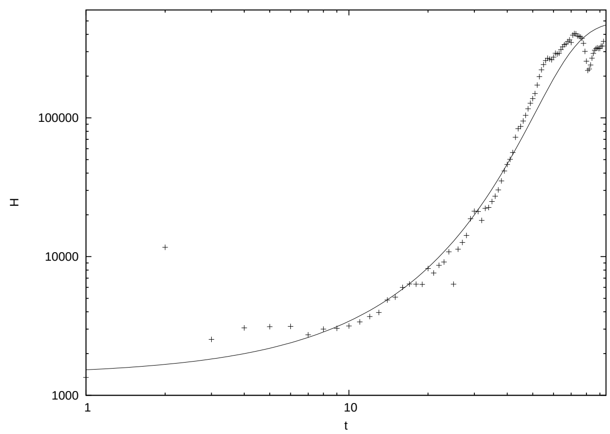

FIG. 4: The growth of the human resource strength is fitted globally by the theoretical model for $\alpha = 1$, $\lambda = 0.09$ and $\eta = 2 \times 10^{-6}$. There has been a noticeable depletion of human resource on the same nonlinear saturation time scale given by Eq. (4), i.e. $75 - 80\,\mathrm{years}$.

differential equation devised by Verhulst to study population dynamics [2, 3]. This equation has also been applied satisfactorily to a wide range of other cases involving growth [2]. That the industrial growth model, based on the *IBM* data, should also fall within the same category is worthy of special note.

The time scale for the onset of nonlinearity can be determined by requiring the two terms on the right hand side of Eq. (2) to be in rough equipartition with each other. This will give the nonlinear time scale,

$$t_{\mathrm{nl}} \sim -\left(\alpha\lambda\right)^{-1} \ln \left|\eta c^{\alpha}\right| , \tag{4}$$

from which, making use of the values of α, η and c needed to calibrate the *IBM* revenue data (both the annual revenue and the cumulative revenue), one gets $t_{\mathrm{nl}} \sim 75 - 80\,\mathrm{years}$. It is easy to see that this is the same time scale on which nonlinearity causes the onset of saturation in the growth of the company, and an indirect confirmation about the validity of this time scale comes from the plot in Fig. 3, which shows the growth of the net annual earnings of *IBM* (labelled as P, the profit), against time, t. The company suffered major reverses in its net earnings (upto 8 billion dollars in 1993) around 1991-1993, which was indeed very close to 80 years of the company, since its inception in 1914. This intriguing correspondence between the two time scales, arrived at via two distinctly different paths, is arguably much more than a simple coincidence.

The human resource content of an industrial organisation is also another indicator of its prevailing state. In the case of *IBM*, the data for the human resource of the company have been plotted in Fig. 4. Over the years the growth of human resource has been steady, and has followed the qualitative growth trend of the revenue. However, one may readily notice that just like the net earnings of *IBM*, there has been a sharp decline in the number of human resource around the time scale of $75 - 80\,\mathrm{years}$ of the age of the company. This may be construed as another independent piece of evidence in support of the theoretical estimate of the saturation time scale that Eq. (4) indicates. And going by the values of α and η needed for the model fit in Fig. 4, the maximum possible human resource strength that *IBM* can viably employ is predicted from Eq. (3) to be about $500,000$.

By now it has become evident that there is a strong correlation among the variables by which one may monitor the state of an industrial organisation. And in fact it happens that a complete impression of the true state of affairs may only be conveyed by analysing all the variables involved. The concept of the “Balanced Scorecard” is somewhat related to this principle [8]. The meaningful variables can all be diverse in nature, ranging from the finances of an organisation to its human and technological resources and to the market within which the organisation might be operating. The growth rate of any one of these variables may have a correlated functional dependence on the collective current state of all the pertinent variables.

So if one were to study an industrial organisation whose current state is defined completely by a general set of n variables, like R_{c} (or R_{a}) and H, then the growth rate of the i-th. variable, ϕ_i, will be given by

$$\dot{\phi}_i = \Phi_i \left(\{\phi_j\}\right) , \tag{5}$$

with Φ_i being a general function of all the variables in the set. The whole set can be expressed explicitly by making both i and j run from 1 to n. This will give a set of n first-order differential equations, with each one of them being coupled to all the others, and this entire set will form an autonomous first-order dynamical system in an n-dimensional space.

If an industrial organisation generates enough revenue, it becomes financially viable for it to maintain a sizeable human

resource pool. On the other hand, the human resource strength will translate into a greater ability to generate revenue. In this manner both the revenue and the human resource content of an organisation will sustain the growth of each other. Defining a general revenue variable, R (which can be either R_{a} or R_{c}), its coupled dynamic growth features along with the human resource, H, can be formally stated in mathematical terms as

$$\begin{aligned} \dot{R} &= \rho\,(R, H) \\ \dot{H} &= \sigma\,(R, H)\ . \end{aligned} \tag{6}$$

The foregoing coupled set of autonomous first-order differential equations forms a two-dimensional system involving two variables, R and H. The equilibrium condition of this dynamical system is obtained when both the derivatives on the left hand side of Eqs. (6) vanish simultaneously, i.e. $\dot{R} = \dot{H} = 0$. The corresponding coordinates in the H—R plane may be labelled (H_0, R_0). Since the terminal state implies the cessation of all growth in time (i.e. all derivatives with respect to time will vanish), it is now possible to argue that the equilibrium state in the H—R plane actually represents a terminal state in real time growth.

Some general deductions can now be made about the nature of the equilibrium state, with the help of dynamical systems theory [3, 4]. The two coupled equations, given by Eqs. (6), will, in the most general sense, be nonlinear. A linearisation treatment on them could be carried out by applying small perturbations on R and H about their equilibrium state values. The perturbation scheme will be $R = R_0 + R'$ and $H = H_0 + H'$. This will allow a coupled set of linearised equations to be set down as

$$\begin{aligned} \frac{\mathrm{d}R'}{\mathrm{d}t} &= \mathcal{A}R' + \mathcal{B}H' \\ \frac{\mathrm{d}H'}{\mathrm{d}t} &= \mathcal{C}R' + \mathcal{D}H', \end{aligned} \tag{7}$$

in which

$$\begin{aligned} \mathcal{A} &= \left.\frac{\partial\rho}{\partial R}\right|_{R_0}, \qquad & \mathcal{B} &= \left.\frac{\partial\rho}{\partial H}\right|_{H_0}, \\ \mathcal{C} &= \left.\frac{\partial\sigma}{\partial R}\right|_{R_0}, \qquad & \mathcal{D} &= \left.\frac{\partial\sigma}{\partial H}\right|_{H_0}. \end{aligned} \tag{8}$$

Solutions of the form $R' \sim e^{\omega t}$ and $H' \sim e^{\omega t}$ will enable one to derive the eigenvalues, ω, of the stability matrix implied by Eqs. (7), as

$$\omega = \frac{1}{2}\left[(\mathcal{A}+\mathcal{D}) \pm \sqrt{(\mathcal{A}+\mathcal{D})^2 - 4\,(\mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C})}\right]\ . \tag{9}$$

The exact determination of the values of the two roots of ω will impart a clear idea about the nature of the equilibrium state, which can either be a saddle point or a node or a focus [3, 4]. The last case will necessarily mean an oscillatory nature in the growth of both R and H through time [3, 4], and is not tenable here. Going back to Figs. 1, 2 & 4, one notices

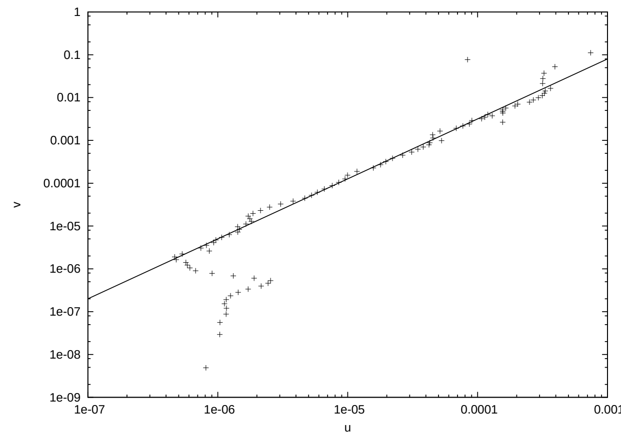

FIG. 5: The straight-line fit validates the logistic equation model. The slope of the straight line is given as 1.4, and it closely matches the value of 1.6 that could be obtained from the parameter fitting in Figs. 2 & 4. The cusp at the bottom left is due to the loss of human resource. The lower arm of the cusp has nearly the same slope as the straight-line above it. Growth, corresponding to the positive slope in this plot, can be modelled well by the logistic equation.

that the respective growth patterns of *both* R and H have, on the other hand, been largely monotonic in nature (except for a manifestly sharp dip in H at large values of t). So an immediate conclusion that follows is that the equilibrium state is very likely a node [3, 4], and this will correspond mathematically to ω having two real roots of the same sign. Practically speaking, this is what is to be expected entirely. For an industrial organisation it is not conceivable that while there is growth in one variable, there will be decay in the other (if this were to happen, the equilibrium state will be a saddle point). Both will have to grow in close mutual association, and, if the dynamical systems argument is anything to go by, both will make the approach towards the terminal state simultaneously. The qualitative behaviour of one variable cannot be completely independent of the other.

To have a quantitative corroboration of the dynamical systems modelling with the help of the *IBM* data, one could, motivated by Eq. (1), proceed with a simple ansatz to describe the individual growth patterns of R and H by an uncoupled logistic equation model. This can be set down as

$$\begin{aligned} \dot{R}(t) &= \lambda_r R\,(1 - \eta_r R^{\alpha_r}) \\ \dot{H}(t) &= \lambda_h H\,(1 - \eta_h H^{\alpha_h})\ , \end{aligned} \tag{10}$$

with the subscripts r and h in the parameters α, λ and η, indicating that R and H will each, in general, have its own different set of parameter values. From Eqs. (10) one could easily eliminate the two time derivatives and arrive at

$$\frac{\mathrm{d}R}{\mathrm{d}H} = \frac{\lambda_r R\,(1 - \eta_r R^{\alpha_r})}{\lambda_h H\,(1 - \eta_h H^{\alpha_h})}\,, \tag{11}$$

whose integral solution can be expressed in a compact power-

law form as

$$v = \kappa u^{\beta} , \tag{12}$$

under the definitions that

$$v = \frac{1}{R^{\alpha_r}} - \eta_r , \qquad u = \frac{1}{H^{\alpha_h}} - \eta_h , \qquad \beta = \frac{\alpha_r \lambda_r}{\alpha_h \lambda_h} , \tag{13}$$

and κ is an integration constant. The power law in Eq. (12) implies that a $\log$-$\log$ plot of v against u will be a straight line with a slope, β. For the greater part of it, this fact has been established emphatically in Fig. 5. In this plot, v has been defined in terms of the cumulative revenue, i.e. $R = R_{\mathrm{c}}$, while going by the model fitting in Figs. 1, 2 & 4, the values $\alpha_r = \alpha_h = 1$, $\eta_r = 5 \times 10^{-7}$ and $\eta_h = 2 \times 10^{-6}$ have been used. The cusp in the bottom left corner of the plot has arisen because of an irregular depletion of human resource in *IBM* in the early 1990s, and hence it is beyond the scope of the logistic equation model. However, the lower arm of the cusp has nearly the same positive slope as the straight-line fit. So, as long as there is steady growth in both R and H, the logistic equation model gives a good description of the global performance of an industrial organisation. A company can be said to be performing well if it is on a curve with a positive slope in the u—v plot, as it has been implied in Fig. 5. Sustained deviations from the straight line and having a negative slope in this $\log$-$\log$ plot, are causes for worry regarding the well-being of the company. The slope of the straight line in Fig. 5 sets a value for the power-law exponent in Eq. (12) as $\beta \simeq 1.4$, which is quite close to the value of $\beta \simeq 1.6$, found simply by taking the ratio of the respective theoretical values of λ, chosen to fit the empirical data in Figs. 2 & 4.

It is remarkable that the simple logistic equation employed for the dynamical system in Eqs. (10), would suffice to model the *IBM* data so closely, as Fig. 5 shows in no uncertain a manner. Once it is obvious that the dynamical systems modelling has been successful, one could formulate a precise description of the terminal state. This can be done by referring to Eqs. (8) and (9), for which, going by Eqs. (10), it is easy to see that neither does ρ depend on H, nor does σ depend on R. Consequently, one has $\mathcal{B} = \mathcal{C} = 0$. Immediately the two roots of Eq. (9) are found to be $\omega = \mathcal{A}$ and $\omega = \mathcal{D}$. To ascertain the exact values of $\mathcal{A}$ and $\mathcal{D}$ in terms of the known parameters, one will have to know the equilibrium values of R and H. These can be shown to be $R_0 = \eta_r^{-1}$ and $H_0 = \eta_h^{-1}$, corresponding to $u = v = 0$ for the terminal state. After this, it is straightforward to argue that $\mathcal{A} = -\lambda_r$ and $\mathcal{B} = -\lambda_h$. Since the values of λ are positive in all the cases of model fitting, it is now possible to claim that with both the roots of ω being real negative numbers, the limiting state for industrial growth is represented by a stable node in the phase portrait of an autonomous first-order dynamical system [4]. Extending this contention further, the limiting state can be perceived to be an attractor state, towards which there will be an asymptotic approach through an infinite passage of time [4]. At least this is what the growth data pertaining to *IBM* indicate.

The terminal feature in industrial growth derives from the nonlinear term in the general logistic equation used for the modelling. This will naturally suggest that the retarding factors acting against the growth of an organisation are nonlinear in character. These factors are all quantified through the parameters α and η in Eq. (1). Of these two parameters, η is more amenable to manipulations, and conditions conducive to lasting growth will necessitate tuning it down. This ought to be the guiding principle behind a successful management strategy for feasible long-term growth, especially in the case of organisations that are still in their early stages. Knowledge of the general nature of the possible adversities lying ahead, can enable a company to apply appropriately corrective measures at the right juncture, and defer the arrival of the nonlinear time scale. As a result it will make a more effective implementation of future strategies and innovative solutions for growth, all of which should be in a state of adaptable alignment with their objectives and core competencies. The "Blue Ocean" strategy [9], for instance, might be one such practical and effective means.

On the other hand, it would never be too late for companies which would already have entered a terminal phase, to institute proper reality checks. This will allow for a more functional and timely redefining of fundamental objectives. The solutions which might follow, could be varied in many unexpected ways. Rather than adhering to conventional modes of growth and survival, industries could devise ways of preserving both their existence and their relevance by being more integrated with the social welfare of their markets — in short, assist in creating an environment of overall prosperity and a general feeling of well-being, right alongside the creation of wealth. The benefits derived through such inclusive strategy implementation should be lasting.

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* Electronic address: akr@hbcse.tifr.res.in

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